Expected Value Theory II: Risk, Uncertainty, Infinity

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Risk: The Allais Paradox

This problem comes from the French economist Maurice Allais, who raised it as a counterexample to Leonard Savage's *Sure-Thing Principle* (which is one of the central axioms underlying Expected Utility Theory).

Roughly, the principle says: if two gambles agree on what happens if one event obtains ($\neg E$), then your ranking of them should depend only on how you rank what would happen if this event doesn't obtain (E).

Sure-Thing Principle

	E	$\neg E$
f	X	Z
g	Y	Z
f^*	X	Z^*
g^*	Y	Z^*

$f \succ g$ if and only if $f^* \succ g^*$

Consider the following two lotteries:

(1_A) a 11% chance of winning $1,000,000.

(1_B) a 10% chance of winning $5,000,000.

Which do you prefer? ____________

Now consider to other lotteries:

(2_A) a 100% chance of winning $1,000,000.

(2_B) a 10% chance of winning $5,000,000, and a 89% chance of winning $1,000,000.

Which do you prefer? ____________

Question: Can you assign utilities to $0, $1,000,000, and $5,000,000 so that your ranking of the lotteries obey Expected Utility Theory?

The Allais Preferences: $1_B \succ 1_A$, $2_A \succ 2_B$. If you have the Allais Preferences, the answer is: *No*.

Is this, then, a counterexample to Expected Utility Theory?

The Allais Paradox

	Tickets 1	2–11	12–100
1_A	$1,000,000	$1,000,000	$0
1_B	$0	$5,000,000	$0
2_A	$1,000,000	$1,000,000	$1,000,000
2_B	$0	$5,000,000	$1,000,000

Uncertainty: The Ellsberg Paradox

This problem comes from Daniel Ellsberg, which is another counterexample to Savage's *Sure-Thing Principle*.

The example turns, not on risk, but on *uncertainty*. (The phenomenon is sometimes called *ambiguity aversion*.)

Ellsberg worked as a U.S. military analyst—specializing in nuclear war policy. Famously, in 1971, Ellsberg (while working for the RAND Corporation) leaked the *Pentagon Papers*.

There are 90 marbles in an urn: 30 are red, the rest are either white or black (in some proportion, which is not known). A marble will be drawn at random. Consider the following gambles:

(1_E) If the marble is red, win $100; otherwise, nothing.

(1_F) If the marble is white, win $100; otherwise, nothing.

Which do you prefer? ____________

(2_E) If the marble is either red or white, win $100; otherwise, nothing.

(2_F) If the marble is white or black, win $100; otherwise, nothing.

Which do you prefer? ____________

The Ellsberg Paradox

	Red	*White*	*Black*
1_E	$100	$0	$0
1_F	$0	$100	$0
2_E	$100	$0	$100
2_F	$0	$100	$100

The Ellsberg Preferences: $1_E \succ 1_F$, $2_F \succ 2_E$.

Infinity: Pascal's Wager

Believing in God has higher *expected value* than not believing, so you should choose to be a Believer.

PASCAL'S WAGER (EXPECTED VALUE)

	God Exists	*God Doesn't Exist*
B	∞	f_1
$\neg$**B**	f_2	f_3

Expected Value of Believing:

$$EU(\mathbf{B}) = \Pr(\text{God Is}) \cdot \infty + \Pr(\text{God Isn't}) \cdot f_1 = \infty$$

Expected Value of Not Believing:

$$EU(\neg\mathbf{B}) = \Pr(\text{God Is}) \cdot f_2 + \Pr(\text{God Isn't}) \cdot f_3 = \text{finite}$$

∞ is (obviously) larger than any finite value, so $EU(\mathbf{B}) > EU(\neg\mathbf{B})$. So, you should choose to believe in God.

(1) Rationality requires that you assign some positive probability to *God Exists.* (2) If you assign some positive probability to *God Exists*, then believing (**B**) has higher expected value than not believing ($\neg$**B**). (3) You should maximize expected value.

Note: The argument give us a *practical* reason to believe in God, not an *epistemic* reason.

Problem of Mixed Strategies. We have more options than **B** and $\neg$**B**; we could employ a *mixed strategy:* e.g., flip a coin, and believe if heads, disbelieve if tails.

But *anything* you might choose to do could be consider a mixed strategy between the two, so *everything* has ∞ value! So it's permissible to do anything!(?)

COIN BET

	Heads	*Tails*
Sure-Thing	∞	∞
Bet on Heads	∞	0

Intuitively, **Sure-Thing** is better than **Bet on Heads**, but they have the same expected value.

Many Gods Objection. Pascal argues that we are rationally required to believe in God. But which one?

MANY GODS WAGER

	Generous God	*Rewarding God*	*Weird God*	*No God*
B	∞	∞	f_2	f_1
$\neg$**B**	∞	f_2	∞	f_3

$$EU(\mathbf{B}) = \Pr(\text{GG}) \cdot \infty + \Pr(\text{RG}) \cdot \infty + \Pr(\text{WG}) \cdot f_2 + \Pr(\text{No God}) \cdot f_1$$
$$EU(\neg\mathbf{B}) = \Pr(\text{GG}) \cdot \infty + \Pr(\text{RG}) \cdot f_2 + \Pr(\text{WG}) \cdot \infty + \Pr(\text{No God}) \cdot f_3$$

Proposal for Dealing with Infinities: Replace each '∞' with a variable N. If there is some $n = N$, such that for all $n^* \geq n,\ EU_N(\phi) > EU_N(\psi)$, for all ψ, then you ought to choose ϕ.

This is a proposal suggested by Caspar Hare.

$$EU_N(\mathbf{B}) = \Pr(\text{GG}) \cdot N + \Pr(\text{RG}) \cdot N + \Pr(\text{WG}) \cdot f_2 + \Pr(\text{No God}) \cdot f_1$$
$$EU_N(\neg\mathbf{B}) = \Pr(\text{GG}) \cdot N + \Pr(\text{RG}) \cdot f_2 + \Pr(\text{WG}) \cdot N + \Pr(\text{No God}) \cdot f_3$$

And, there is some $n = N$, such that for all $n^* \geq n,\ EU_N(\mathbf{B}) > EU_N(\neg\mathbf{B})$ *only if* $\Pr(RG) > \Pr(WG)$: only if you are antecedently more confident that there is a god who rewards all and only Believers than you are that there is a god who rewards all and only Non-believers.

Proof. $EU_N(\mathbf{B}) > EU_N(\neg\mathbf{B})$ if and only if $\Pr(\text{GG})(N - N) + \Pr(\text{RG})(N - f_2) + \Pr(\text{WG})(f_2 - N) + \Pr(\text{No God})(f_1 - f_3) > 0$.

Which holds just in case:

$$\Pr(\text{RG}) \cdot (N - f_2) > \Pr(\text{WG}) \cdot (N - f_2) + \Pr(\text{No God}) \cdot (f_3 - f_2)$$
$$\Pr(\text{RG}) > \Pr(\text{WG}) \cdot \frac{N - f_2}{N - f_2} + \Pr(\text{No God}) \cdot \frac{f_3 - f_2}{N - f_2}$$
$$\Pr(\text{RG}) > \Pr(\text{WG}) + \Pr(\text{No God}) \cdot \frac{f_3 - f_2}{N - f_2}$$

As $N \to \infty,\ \Pr(\text{No God}) \cdot \frac{f_3 - f_2}{N - f_2} = 0$. So, $EU_N(\mathbf{B}) > EU_N(\neg\mathbf{B})$ only if $\Pr(\text{RG}) > \Pr(\text{WG})$.